

Applications of Symmetric Functions to Cycle and Increasing Subsequence Structure after
Shuffles (Part 2)

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Abstract

Using the Berele/Remmel/Kerov/Vershik variation of the Robinson-Schensted-Knuth correspondence, we study the cycle and increasing subsequence structure after various methods of shuffling. One consequence is a cycle index for shuffles like: cut a deck into two roughly equal piles, thoroughly mix the first pile and then riffle it with the second pile. Conclusions are drawn concerning the distribution of fixed points and the asymptotic distribution of cycle structure. An upper bound on the convergence rate is given. Connections are made with extended Schur functions and with point process work of Baik and Rains.

Keywords: Card shuffling, Robinson-Schensted-Knuth correspondence, cycle index, increasing subsequence, random matrix.

1 Introduction

In an unpublished effort to study the way real people shuffle cards, Gilbert-Shannon-Reeds introduced the following model, called k -riffle shuffling. Given a deck of n cards, one cuts it into k piles with probability of pile sizes j_1, \dots, j_k given by $\frac{\binom{n}{j_1, \dots, j_k}}{k^n}$. Then cards are dropped from the packets with probability proportional to the pile size at a given time (thus if the current pile sizes are A_1, \dots, A_k , the next card is dropped from pile i with probability $\frac{A_i}{A_1 + \dots + A_k}$). A celebrated result of Bayer and Diaconis [BayD] is that $\frac{3}{2} \log_2(n)$ two-shuffles are necessary and suffice for randomness.

One of the most remarkable properties of GSR k -shuffles is the following. Since k -shuffles induce a probability measure on conjugacy classes of S_n , they induce a probability measure on partitions of n . Consider the factorization of a random degree n polynomial over a finite field F_q into irreducibles. The degrees of the irreducible factors of a randomly chosen degree n polynomial also give a random partition of n . The fundamental result of Diaconis-McGrath-Pitman (DMP) [DMP] is that this measure on partitions of n agrees with the measure induced by card shuffling when $k = q$. This allowed natural questions on shuffling to be reduced to known results on factors of polynomials and vice versa.

The DMP theorem is deep and has connections to many parts of mathematics (e.g. Hochschild homology [H], dynamical systems [La], and Lie theory [F2],[F3],[F4]). Stanley [St1] gave a proof of the DMP theorem using ideas from symmetric function theory. He also related the Robinson-Schensted-Knuth shape of a permutation after a shuffle to Schur functions and gave connections with work of the random matrix community. Following the appearance of [St1], the paper [F5] gave a different symmetric function theoretic proof of the DMP result. Although more complicated than Stanley's proof (in particular it needed the RSK algorithm), it had the merit of suggesting natural extensions of the DMP result.

This note is a continuation of [F5] (Part 1) and was motivated by an effort to understand the relation between it and the paper [KV], in particular the fact that the S_λ used in Part 1 are extended Schur functions. Section 2 defines models of card shuffling called $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffles which include the GSR shuffles. This model contains other shuffles of interest such as: given a deck of n cards, cut off $\text{binomial}(n, 1/2)$ many cards as in the GSR 2-shuffle, shuffle them thoroughly, and then riffle them with the remaining cards (this special case was first studied in [DFP]). It is proved that if one applies the RSK correspondence to a permutation distributed as a $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffle, then the probability of getting any recording tableau of shape λ is the extended Schur function $S_\lambda(\vec{\alpha}, \vec{\beta}, \gamma)$.

For the case $\gamma = 0$ this result will be reduced to Proposition 3 of the paper [KV]. Since Proposition 3 of [KV] is incorrect for $\gamma \neq 0$, Section 2 undertakes the repair needed for the applications to card shuffling. Section 3 proves Cauchy-type identities for the extended Schur functions. (The method of proof closely follows that of [TrWid] for Schur functions; since we need the Cauchy type identities and similar reasoning later in the paper, we include the details). Section 4 connects the shuffling models with work of Baik and Rains [BaiRa] on increasing subsequences for point process models. The final section applies the work of earlier sections to find formulas for cycle structure after $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffles; for instance it is proved that after such a shuffle on a deck of size n , the expected number of fixed points is the sum of the first n extended power sum symmetric functions evaluated at the relevant parameters. An upper bound on the convergence rate of the shuffles is derived.

As Stanley [St1] notes, the results about longest increasing subsequences after riffle shuffles (which is the same as longest weakly increasing subsequences in random words) tie in with work of the random matrix community (e.g. [TrWid],[Ku],[J]). We expect that the connection between card shuffling and random matrix theory holds for all of the shuffles considered here (and possibly for the shuffles in [F3],[F4]). Some work has been done in this direction (not in the language of card shuffling) and is nicely surveyed in [BorO]. It is remarkable that the unified viewpoint of [BorO] uses the representation theory of the infinite symmetric group (extended Schur functions), which are also the parameterizing data in this paper.

2 Extended Schur Functions and the RSK Correspondence

The extended complete symmetric functions $\tilde{h}_k(\alpha, \beta, \gamma)$ are defined by the generating function

$$\sum_{k=0}^{\infty} \tilde{h}_k(\alpha, \beta, \gamma) z^k = e^{\gamma z} \prod_{i \geq 1} \frac{1 + \beta_i z}{1 - \alpha_i z}.$$

For $\lambda = (\lambda_1, \dots, \lambda_n)$, the extended Schur functions are defined by

$$\tilde{s}_\lambda = \det(\tilde{h}_{\lambda_i - i + j})_{i,j=1}^n.$$

Since the extended Schur functions give the characters of the infinite symmetric group, they are very natural objects.

Thoma proves the following result.

Theorem 1 (*[Th]*) *Let $G(z) = \sum_{k=0}^{\infty} g_k z^k$ be such that $g_0 = 1$ and all $g_k \geq 0$. Then*

$$\det(g_{\lambda_i - i + j})_{i,j=1}^n \geq 0$$

for all partitions λ if and only if

$$G(z) = e^{\gamma z} \prod_{i \geq 1} \frac{1 + \beta_i z}{1 - \alpha_i z}$$

where $\gamma \geq 0$ and $\sum \beta_i, \sum \alpha_i$ are convergent series of positive numbers.

Given Thoma's result, it is natural to interpret the extended Schur functions as probabilities. This is the topic of the paper [KV]; however their Proposition 3 is false for $\gamma \neq 0$. This section repairs their Proposition 3 and makes a connection with card shuffling.

We suppose that $\gamma + \sum \alpha_i + \sum \beta_i = 1$ and that $\gamma \geq 0, \alpha_i, \beta_i \geq 0$ for all i . Using these parameters, we define a random permutation on n symbols as follows. First, create a word of length n by choosing letters n times independently according to the rule that one picks $i > 0$ with probability α_i , $i < 0$ with probability β_i , and $i = 0$ with probability γ . We use the usual ordering $\dots < -1 < 0 < 1 < \dots$ on the integers. Starting with the smallest negative symbol which appears in the word, let m be the number of times it appears. Then write $\{1, 2, \dots, m\}$ under its appearances in *decreasing* order from left to write. If the next negative symbol appears k times write $\{m+1, \dots, m+k\}$ under its appearances, again in decreasing order from left to write. After finishing with the negative symbols, proceed to the 0's. Letting r be the number of 0's, choose a random permutation of the relevant r consecutive integers and write it under the 0's. Finally, move to the positive symbols. Supposing that the smallest positive symbol appears s times, write the relevant s consecutive integers under its appearances in *increasing* order from left to right.

The best way to understand this procedure is through an example. Given the string

$$-2 \ 0 \ 1 \ 0 \ 0 \ 2 \ -1 \ -2 \ -1 \ 1$$

one obtains each of the six permutations

$$2 \ 5 \ 8 \ 6 \ 7 \ 10 \ 4 \ 1 \ 3 \ 9$$

$$2 \ 5 \ 8 \ 7 \ 6 \ 10 \ 4 \ 1 \ 3 \ 9$$

$$2 \ 6 \ 8 \ 5 \ 7 \ 10 \ 4 \ 1 \ 3 \ 9$$

$$2 \ 6 \ 8 \ 7 \ 5 \ 10 \ 4 \ 1 \ 3 \ 9$$

2 7 8 5 6 10 4 1 3 9

2 7 8 6 5 10 4 1 3 9

with probability $1/6$. In all cases the 1, 2 correspond to the -2 's, the 3, 4 correspond to the -1 's, the 8, 9 correspond to the 1's and the 10 corresponds to the 2. The symbols 5, 6, 7 correspond to the 0's and there are six possible permutations of these symbols. We call this probability measure on permutations a $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffle.

The following elementary result (generalizing results in [BayD] and [DFP]) gives physical descriptions of these shuffles and explains how they convolve. The proof method follows that of [BayD].

Proposition 1 1. A $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffle is equivalent to the following procedure. Cut the n card deck into piles with sizes X_i indexed by the integers, where the probability of having $X_i = x_i$ for all i is equal to

$$\frac{n!}{\prod_{i=-\infty}^{\infty} x_i!} \gamma^{x_0} \prod_{i>0} \alpha_i^{x_i} \prod_{i<0} \beta_i^{x_i}.$$

The top cards go to the non-empty pile with smallest index, the next batch of cards goes to the pile with second smallest index, and so on. Then mix the pile indexed by 0 until it is a random permutation, and turn upside down all of the piles with negative indices. Finally, riffle the piles together as in the first paragraph of the introduction and look at the underlying permutation (i.e. ignore the fact that some cards are upside down).

2. The inverse of a $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffle is equivalent to the following procedure. Randomly label each card of the deck, picking label 0 with probability γ , label $i > 0$ with probability α_i and label $i < 0$ with probability β_i . Deal cards into piles indexed by the labels, where cards with negative or zero label are dealt face down and cards with positive label are dealt face up. Then mix the pile labelled 0 so that it is a random permutation and turn all of the face up piles face down. Finally pick up the piles by keeping piles with smaller labels on top.

3. Performing a $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffle k times is the same as performing the following shuffle. One cuts into piles with labels given by k -tuples of integers (z_1, \dots, z_k) ordered according to the following rule:

(a) $(z_1, \dots, z_k) < (z'_1, \dots, z'_k)$ if $z_1 < z'_1$.

(b) $(z_1, \dots, z_k) < (z'_1, \dots, z'_k)$ if $z_1 = z'_1 \geq 0$ and $(z_2, \dots, z_k) < (z'_2, \dots, z'_k)$.

(c) $(z_1, \dots, z_k) < (z'_1, \dots, z'_k)$ if $z_1 = z'_1 < 0$ and $(z_2, \dots, z_k) > (z'_2, \dots, z'_k)$.

The pile is assigned probability equal to the product of the probabilities of the symbols in the k tuple. Then the shuffle proceeds as in part 1, where negative piles (piles where the product of the coordinates of the k tuple are negative) are turned upside down and piles with some coordinate equal to 0 are perfectly mixed before the piles are all riffled together.

Examples As an example of Proposition 1, consider an $(\alpha_1, \alpha_2; \beta_1, \beta_2; \gamma)$ shuffle with $n = 11$. For part 1, it may turn out that $X_{-2} = 2$, $X_{-1} = 1$, $X_0 = 3$, $X_1 = 2$, and $X_2 = 3$. Then the deck is cut into piles $\{1, 2\}$, $\{3\}$, $\{4, 5, 6\}$, $\{7, 8\}$, $\{9, 10, 11\}$. The first two piles are turned upside down and the third pile is completely randomized, which might yield piles $\{2, 1\}$, $\{3\}$, $\{5, 4, 6\}$, $\{7, 8\}$, $\{9, 10, 11\}$. Then these piles are riffled together as in the GSR shuffle. This might yield the permutation

$$5 \ 2 \ 7 \ 4 \ 8 \ 9 \ 10 \ 3 \ 1 \ 11 \ 6.$$

The inverse description (part 2) would amount to labelling cards 2,9 with -2 , card 8 with -1 , cards 1,4,11 with 0, card 3,5 with 1, and cards 6,7,10 with 2, and then mixing the 0 pile as 4, 1, 11. Note that this leads to the permutation (inverse to the previous permutation)

$$9 \ 2 \ 8 \ 4 \ 1 \ 11 \ 3 \ 5 \ 6 \ 7 \ 10.$$

As an example of part 3, note that doing a $(\alpha_1; \beta_1; 0)$ shuffle twice does not give a $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffle, but rather gives a shuffle with 4 piles in the order $(-1, 1), (-1, -1), (1, -1), (1, 1)$ where pile 1 has probability $\beta_1 \alpha_1$, pile 2 has probability $\beta_1 \beta_1$, pile 3 has probability $\alpha_1 \beta_1$ and pile 4 has probability $\alpha_1 \alpha_1$. Piles 1 and 3 are turned upside down before the riffing takes place. From part 1 of this paper one can still analyze the cycle structure and RSK shape of these shuffles even though they aren't $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffles. (Actually part 1 of this paper looks at shuffles conjugate to these shuffles by the longest element; this clearly has no effect on the cycle index and has no effect on the RSK shape by a result of Schützenberger exposted as Theorem A1.2.10 in [St2]).

As another example of part 3, note that a shuffle with parameters $(\alpha_1; 0; \gamma)$ repeated twice gives a shuffle with 4 piles in the order $(0, 0), (0, 1), (1, 0), (1, 1)$ where the first 3 piles are completely mixed before all piles are riffled together. This is clearly the same as a $(\alpha_1^2; 0; 1 - \alpha_1^2)$ shuffle, agreeing with Lemma 2.1 of [DFP].

Berele and Remmel [BeRe] and independently Kerov and Vershik [KV] consider the following analog of the RSK correspondence. Given a word on the symbols $\{\pm 1, \pm 2, \dots\}$ one runs the RSK correspondence with the ammendments that a negative symbol is allowed to bump itself, but that a positive symbol can't bump itself. For example the word

$$1 \ -1 \ 2 \ -2 \ 1 \ 1 \ -2$$

has insertion tableau P and recording tableau Q respectively equal to

$$\begin{array}{ccc} -2 & 1 & 1 \\ -2 & 2 & \\ -1 & & \\ 1 & & \\ & & \\ & & \\ & & \\ & & \\ & & \end{array} \quad \begin{array}{ccc} 1 & 3 & 6 \\ 2 & 5 & \\ 4 & & \\ 7 & & \end{array}$$

Theorem 2 ([BeRe],[KV]) *The above variation on the Robinson-Schensted-Knuth correspondence gives a bijection between words of length n from the alphabet of integers with the symbol i appearing n_i times and pairs (P, Q) where*

1. *The symbol i occurs n_i times in P .*
2. *The entries of P are weakly increasing in rows and columns.*
3. *Each positive symbol occurs at most once in each column of P and each negative symbol occurs at most once in each row of P .*
4. *Q is a standard Young tableau on the symbols $\{1, \dots, n\}$.*

Furthermore,

$$S_\lambda(\vec{\alpha}, \vec{\beta}, 0) = \sum_{\substack{P \\ \text{shape}(P)=\lambda}} \prod_{i>0} \alpha_i^{n_i(P)} \prod_{i<0} \beta_i^{n_i(P)}.$$

Theorem 3 and Corollary 1 connect card shuffling to the extended Schur functions. Related results can be found [St1] and [F5] (in particular, Corollary 1 was proved in [St1] for usual Schur functions).

Theorem 3 *Let π be distributed as a permutation under a $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffle. Let Q be any standard Young tableau of shape λ . Then the probability that π has Robinson-Schensted-Knuth recording tableau equal to Q is $S_\lambda(\vec{\alpha}, \vec{\beta}, \gamma)$.*

PROOF: First suppose that $\gamma = 0$. As indicated earlier in this section, each length n word w on the symbols $\{\pm 1, \pm 2, \dots\}$ defines a permutation π . From this construction, it is easy to see that the recording tableau of w under the BRKV variation of the RSK algorithm is equal to the recording tableau of π under the RSK algorithm. Thus it is enough to prove that the probability that the word w has BRKV recording tableau Q is $S_\lambda(\vec{\alpha}, \vec{\beta}, 0)$. This is immediate from Theorem 2.

Now the case $\gamma \neq 0$ can be handled by introducing m extra symbols between 0 and 1—call them $1/(m+1), 2/(m+1), \dots, m/(m+1)$ and choosing each with probability γ/m . Thus the random word is on $\{\pm 1, \pm 2, \dots\}$ and these extra symbols. Each word defines exactly one permutation—the symbols $1/(m+1), 2/(m+1), \dots, m/(m+1)$ are treated as positive. By the previous paragraph, the probability of obtaining recording tableau Q is equal to $S_\lambda(\vec{\alpha}, \vec{\beta})$ where the associated \tilde{h}_k are defined by

$$\sum_{k=0}^{\infty} \tilde{h}_k(\alpha, \beta) z^k = \left(\frac{1}{1 - \gamma z/m} \right)^m \prod_{i \geq 1} \frac{1 + \beta_i z}{1 - \alpha_i z}.$$

As $m \rightarrow \infty$, this distribution on permutations converges to that of a $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffle, and the generating function of the \tilde{h}_k converges to

$$\sum_{k=0}^{\infty} \tilde{h}_k(\alpha, \beta, \gamma) z^k = e^{\gamma z} \prod_{i \geq 1} \frac{1 + \beta_i z}{1 - \alpha_i z}.$$

□

Corollary 1 *Let f_λ be the number of standard Young tableau of shape λ . Let π be distributed as a permutation under a $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffle. Then the probability that π has Robinson-Schensted-Knuth shape λ is equal to $f_\lambda S_\lambda(\vec{\alpha}, \vec{\beta}, \gamma)$.*

3 Gessel's Theorem, Szego's Theorem, and Cauchy Identities

Letting $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$ be a function on the unit circle, the Toeplitz determinant $D_n(f)$ is defined as the determinant of the $n \times n$ Toeplitz matrix

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{-1} & c_0 & \cdots & c_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{-(n-1)} & c_{-(n-2)} & \cdots & c_0 \end{pmatrix}.$$

The function f is called the symbol of the Toeplitz determinant. The strong Szego theorem for Toeplitz determinants [BotSi] states that under mild conditions on the symbol,

$$D_n(e^{\sum_{-\infty}^{\infty} c_k z^k}) \sim e^{nc_0 + \sum_{k=1}^{\infty} k c_k c_{-k}}.$$

The aim of this section is to prove the following two identities:

$$\sum_{\substack{\lambda \\ l(\lambda) \leq n}} S_\lambda(\alpha, \beta, \gamma) s_\lambda(x) = D_n \left(e^{\gamma z} \prod_{r=1}^{\infty} \frac{1 + \beta_r z}{(1 - x_r/z)(1 - \alpha_r z)} \right)$$

$$\sum_{\lambda} S_\lambda(\alpha, \beta, \gamma) s_\lambda(x) = \sum_{\lambda} \frac{1}{z_\lambda} \tilde{p}_\lambda(\alpha, \beta, \gamma) p_\lambda(x)$$

The first identity will be proved using a technique of Gessel [G], and the second identity will follow from the first by applying the strong Szego limit theorem for Toeplitz determinants (an idea used in Tracy and Widom [TrWid]).

Theorem 4

$$\sum_{\substack{\lambda \\ l(\lambda) \leq n}} S_\lambda(\alpha, \beta, \gamma) s_\lambda(x) = D_n \left(e^{\gamma z} \prod_{r=1}^{\infty} \frac{1 + \beta_r z}{(1 - x_r/z)(1 - \alpha_r z)} \right)$$

PROOF: Let A be the $n \times \infty$ matrix $(\tilde{h}_{j-i}(x))$ ($i \geq 1, 1 \leq j \leq n$). Let B be the $\infty \times n$ matrix $(h_{i-j}(x))$ ($i \geq 1, 1 \leq j \leq n$). We evaluate in two ways $\det(AB)$.

One on hand,

$$\det(AB) = \det \left(\sum_{k=0}^{\infty} \tilde{h}_{k-i}(\alpha, \beta, \gamma) h_{k-j}(x) \right).$$

Clearly this is a Toeplitz determinant. The symbol is

$$\begin{aligned} \sum_{j=-\infty}^{\infty} z^j \sum_{k=0}^{\infty} \tilde{h}_k(\alpha, \beta, \gamma) h_{k-j}(x) &= \sum_{k=0}^{\infty} \tilde{h}_k(\alpha, \beta, \gamma) \sum_{j=-\infty}^{\infty} z^j h_{k-j}(x) \\ &= \sum_{k=0}^{\infty} z^k \tilde{h}_k(\alpha, \beta, \gamma) \sum_{j=-\infty}^{\infty} z^{j-k} h_{k-j}(x) \\ &= e^{\gamma z} \prod_{r=1}^{\infty} \frac{1 + \beta_r z}{(1 - x_r/z)(1 - \alpha_r z)}. \end{aligned}$$

On the other hand, the Cauchy-Binet formula gives that

$$\det(AB) = \sum_{S=s_1 < \dots < s_n} \det(A|S) \det(S|B)$$

where $(A|S)$ (resp. $(B|S)$) is the $n \times n$ matrix formed by using the columns (resp. rows) indexed by S . Writing $S = \{\lambda_{n+1-i} + i\}$ associates the subsets S with partitions with at most n parts. Then

$$\sum_{S=s_1 < \dots < s_n} \det(A|S) \det(S|B) = \sum_{\substack{\lambda \\ l(\lambda) \leq n}} S_\lambda(\alpha, \beta, \gamma) s_\lambda(x)$$

as desired. \square

We also use the well known Polya cycle index.

Lemma 1 *Let $m_i(\lambda)$ be the number of parts of size i in λ . Then*

$$\sum_{\lambda} \frac{1}{z_\lambda} \prod_i x_i^{m_i(\lambda)} = e^{\sum_{k \geq 1} \frac{x_k}{k}}.$$

PROOF: The coefficient of $\prod_i x_i^{m_i(\lambda)}$ on the left hand side is one over the number of permutations on $\sum_i i m_i$ symbols which commute with a permutation with m_i cycles of length i . This is $\frac{1}{\prod_i i^{m_i} m_i!}$ which agrees with the coefficient of $\prod_i x_i^{m_i(\lambda)}$ on the right hand side. \square

Combined with Theorem 4, this allows us to prove

Theorem 5

$$\sum_{\lambda} S_\lambda(\alpha, \beta, \gamma) s_\lambda(x) = \sum_{\lambda} \frac{1}{z_\lambda} \tilde{p}_\lambda(\alpha, \beta, \gamma) p_\lambda(x)$$

PROOF: Let $n \rightarrow \infty$ in both sides of the statement of Theorem 4. Writing

$$D_n \left(e^{\gamma z} \prod_{r=1}^{\infty} \frac{1 + \beta_r z}{(1 - x_r/z)(1 - \alpha_r z)} \right) = D_n \left(e^{\gamma z + \sum_{r \geq 1} \log(1 + \beta_r z) - \log(1 - \alpha_r z) - \log(1 - x_r/z)} \right)$$

and applying the strong Szego theorem gives that the determinant converges to

$$e^{\sum_{k=1}^{\infty} \frac{1}{k} \tilde{p}_k(\alpha, \beta, \gamma) p_k(x)}.$$

However we know that

$$\sum_{\lambda} \frac{1}{z_\lambda} \tilde{p}_\lambda(\alpha, \beta, \gamma) p_\lambda(x) = e^{\sum_{k=1}^{\infty} \frac{1}{k} \tilde{p}_k(\alpha, \beta, \gamma) p_k(x)}$$

because this equation is simply the Polya cycle index (Lemma 1) with x_i replaced by $\tilde{p}_i(\alpha, \beta, \gamma)p_i(x)$.

□

Richard Stanley has pointed out that Theorems 4 and 5 can be deduced from the corresponding results for usual Schur functions as follows. Since the h_k 's are algebraically independent, there is a unique homomorphism (sending 1 to 1) and h_k to $\tilde{h}_k(\alpha, \beta, \gamma)$ and the Jacobi-Trudi identity shows that s_λ maps to \tilde{s}_λ .

4 Connections with Work of Baik and Rains

This section connects card shuffling with work of Baik and Rains [BaiRa]. They study “extended growth models” indexed by parameter sets which we call $(\vec{\alpha}^+, \vec{\beta}^+, \gamma^+)$ and $(\vec{\alpha}^-, \vec{\beta}^-, \gamma^-)$. The case relevant to this paper is $\vec{\alpha}^+ = \vec{\beta}^+ = \vec{0}$. We assume without loss of generality (one can simply rescale γ^+) that $\gamma^- + \sum \alpha_i^- + \sum \beta_i^- = 1$. In this case, which we call $BR(\gamma^+, \vec{\alpha}^-, \vec{\beta}^-, \gamma^-)$, their model becomes the following:

1. On $[0, 1] \times [0, 1]$ choose $Poisson(\gamma^+ \gamma^-)$ i.i.d. uniform points.
2. On $[0, 1] \times i$ ($i \in \{1, 2, \dots\}$) choose $Poisson(\gamma^+ \alpha_i^-)$ i.i.d. uniform points.
3. On $[0, 1] \times i$ ($i \in \{-1, -2, \dots\}$) choose $Poisson(\gamma^+ \beta_i^-)$ i.i.d. uniform points.

They define a sequence of points (x_i, y_i) to be increasing if $x_i \leq x_{i+1}, y_i \leq y_{i+1}$ and

$$y_i = y_{i+1} \implies y_i \geq 0.$$

They associate to their point process a random partition λ with λ_i defined by the property that

$$\sum_{i=1}^l \lambda_i$$

is the size of the longest subsequence of points which is a union of l increasing subsequences.

They find a Toeplitz determinant expression for the probability that $\lambda_1 < k$. Theorem 6 (which is well known for the case of random permutations (i.e. $\vec{\alpha}^- = \vec{\beta}^- = 0$) relates their point process to card shuffling measures on permutations and gives a formula for the chance that their random partition is λ .

Theorem 6 *1. Consider the random partition arising from the $BR(\gamma^+, \vec{\alpha}^-, \vec{\beta}^-, \gamma^-)$ point process. The probability that this partition is equal to λ is the same as the probability that the*

RSK shape of a permutation after a $(\vec{\alpha}^-, \vec{\beta}^-, \gamma^-)$ shuffle on $\text{Poisson}(\gamma^+)$ symbols is equal to λ .

2. More explicitly, letting f_λ be the number of standard Young tableaux of shape λ , this probability is

$$\frac{(\gamma^+)^{|\lambda|} f_\lambda S_\lambda(\vec{\alpha}^-, \vec{\beta}^-, \gamma^-)}{e^{\gamma^+} |\lambda|!}.$$

PROOF: We associate to a realization of the $\text{BR}(\gamma^+, \vec{\alpha}^-, \vec{\beta}^-, \gamma^-)$ point process a random permutation π as follows. First take the deck size to be the number of points (which has distribution $\text{Poisson}(\gamma^+)$). Rank the y coordinates of the points in increasing order, where one breaks ties for negative y coordinates by defining the point with the larger x coordinate to be smaller and breaks ties for positive y coordinates by defining the point with larger x coordinate to be larger. Then $\pi(i)$ is defined as the rank of the y coordinate of the point with the i th smallest x coordinate (with probability one there is no repetition among x coordinates). For example, if the BR point process yields the points

$$(.2, .3), (.3, .5), (.35, -8), (.4, 9), (.45, 9), (.5, 7), (.6, -2), (.7, -8)$$

then the resulting permutation would be (in 2-line form)

$$4 \ 5 \ 2 \ 7 \ 8 \ 6 \ 3 \ 1.$$

It is easy to see that this distribution on permutations is the same as that arising from a $(\vec{\alpha}^-, \vec{\beta}^-, \gamma^-)$ shuffle. The second part follows from the first part and Theorem 3. \square

As a corollary, we obtain another proof of a result of Baik and Rains.

Corollary 2 ([BaiRa]) *Let λ be the partition associated to the $\text{BR}(\gamma^+, \vec{\alpha}^-, \vec{\beta}^-, \gamma^-)$ point process. Then the probability that λ has largest part at most n is equal to the Toeplitz determinant*

$$\frac{1}{e^{\gamma^+}} D_n \left(e^{\gamma^+/z} e^{\gamma^- z} \prod_{r=1}^{\infty} \frac{1 + \beta_r^- z}{1 - \alpha_r^- z} \right).$$

PROOF: By Theorem 6, the sought probability is

$$\frac{1}{e^{\gamma^+}} \sum_{\substack{\lambda \\ \iota(\lambda) \leq n}} \frac{(\gamma^+)^{|\lambda|} f_\lambda}{|\lambda|!} S_\lambda(\vec{\alpha}^-, \vec{\beta}^-, \gamma^-).$$

Writing

$$\frac{(\gamma^+)^{|\lambda|} f_\lambda}{|\lambda|!} = \det \left(\frac{\gamma^+}{(\lambda_i - i + j)!} \right)$$

(page 117 of [Mac]) and

$$S_\lambda(\vec{\alpha}^-, \vec{\beta}^-, \gamma^-) = \det \left(\tilde{h}_{\lambda_i - i + j}(\vec{\alpha}^-, \vec{\beta}^-, \gamma^-) \right),$$

the result now follows by an argument as in Theorem 4. \square

As a final result, we note that certain specializations of Schur measure on partitions (which are of interest to the random matrix community) have a probability interpretation in terms of distributions on permutations. For its statement, recall that a permutation is said to have a descent at position i if $\pi(i) > \pi(i+1)$; maj denotes the major index of a permutation (sum of the positions of the descents) and d denotes the number of descents of a permutation. Finally $\begin{bmatrix} n \\ m \end{bmatrix}_q$ denotes the q binomial coefficient $\frac{(q^n-1)\cdots(q-1)}{(q^m-1)\cdots(q-1)(q^{n-m}-1)\cdots(q-1)}$.

Theorem 7 *Consider the probability measure on partitions of size n which picks λ a partition of n with probability*

$$\frac{1}{Z_n} s_\lambda(1, \frac{1}{p}, \dots, \frac{1}{p^{k-1}}) s_\lambda(1, \frac{1}{q}, \dots, \frac{1}{q^{l-1}})$$

(where Z_n is the normalization constant which can be computed from Cauchy's identity). This is the pushforward under the RSK correspondence of the measure on S_n which picks a permutation π with probability

$$\frac{1}{Z_n} p^{\text{maj}(\pi^{-1})} q^{\text{maj}(\pi)} \begin{bmatrix} k - d(\pi^{-1}) + n - 1 \\ n \end{bmatrix}_p \begin{bmatrix} l - d(\pi) + n - 1 \\ n \end{bmatrix}_q.$$

PROOF: This follows easily from Proposition 7.9.12 of [St2] together with the fact that if π goes to the pair (P, Q) under the RSK correspondence, the the descent set of π is equal to the descent set of Q and the descent set of π^{-1} is equal to the descent set of P (Lemma 7.23.1 of [St2]). \square

Note that for the case of greatest interest ($p = q = 1$), the normalization constant is $\binom{kl+n-1}{n}$.

5 Applications to Card Shuffling: Convergence Rates and Cycle Index

First we derive an upper bound on the convergence rate of $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffles to randomness using strong uniform times as in [DFP] and then [F1]. The separation distance between a probability

$P(\pi)$ and the uniform distribution $U(\pi)$ is defined as $\max_{\pi}(1 - \frac{Q(\pi)}{U(\pi)})$ and gives an upper bound on total variation distance. Examples of the upper bound of Theorem 8 are considered later.

Theorem 8 *The separation distance between k applications of a $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffle and uniform is at most*

$$\binom{n}{2} \left[\sum_i (\alpha_i)^2 + \sum_i (\beta_i)^2 \right]^k.$$

Thus $k = 2 \log \frac{1}{\sum_i (\alpha_i)^2 + \sum_i (\beta_i)^2} n$ steps suffice to get close to the uniform distribution.

PROOF: For each k , let A^k be a random $n \times k$ matrix formed by letting each entry equal $i > 0$ with probability α_i , $i < 0$ with probability β_i , and $i = 0$ with probability γ . Let T be the first time that all rows of A^k containing no zeros are distinct; from the inverse description of $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffles this is a strong uniform time in the sense of Sections 4B-4D of Diaconis [D], since if all cards are cut in piles of size one the permutation resulting after riffing them together is random. The separation distance after k applications of a $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffle is upper bounded by the probability that $T > k$ [AD]. Let V_{ij} be the event that rows i and j of A^k are the same and contain no zeros. The probability that V_{ij} occurs is $[\sum_i (\alpha_i)^2 + \sum_i (\beta_i)^2]^k$. The result follows because

$$\begin{aligned} \text{Prob}(T > k) &= \text{Prob}(\cup_{1 \leq i < j \leq n} V_{ij}) \\ &\leq \sum_{1 \leq i < j \leq n} \text{Prob}(V_{ij}) \\ &= \binom{n}{2} \left[\sum_i (\alpha_i)^2 + \sum_i (\beta_i)^2 \right]^k \end{aligned}$$

□

The results of Section 2 and Section 3 are used to find a cycle index after $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffles.

Recall that

$$\tilde{p}_1(\vec{\alpha}, \vec{\beta}, \gamma) = \sum_i \alpha_i + \sum_i \beta_i + \gamma = 1$$

and (for $n \geq 2$)

$$\tilde{p}_n(\vec{\alpha}, \vec{\beta}, \gamma) = \sum_i (\alpha_i)^n + (-1)^{n+1} \sum_i (\beta_i)^n.$$

Theorem 9 1. Let $E_{n,(\vec{\alpha}, \vec{\beta}, \gamma)}$ denote expected value after a $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffle of an n card deck.

Let $N_i(\pi)$ be the number of i -cycles of a permutation π . Then

$$\sum_{n \geq 0} u^n E_{n,(\vec{\alpha},\vec{\beta},\gamma)} \left(\prod_i x_i^{N_i} \right) = \prod_{i,j} e^{\frac{(u^i x_i)^j}{ij} \sum_{d|i} \mu(d) \tilde{p}_{jd}(\vec{\alpha},\vec{\beta},\gamma)^{i/d}}.$$

2. Let $E'_{n,(\vec{\alpha},\vec{\beta},\gamma)}$ denote expected value after a $(\vec{\alpha},\vec{\beta},\gamma)$ shuffle of an n card deck followed by reversing the order of the cards. Then

$$\sum_{n \geq 0} u^n E'_{n,(\vec{\alpha},\vec{\beta},\gamma)} \left(\prod_i x_i^{N_i} \right) = \sum_{n \geq 0} u^n E_{n,(\vec{\beta},\vec{\alpha},\gamma)} \left(\prod_i x_i^{N_i} \right).$$

PROOF: Given the results of Section 2 and Section 3, the proof of the first part runs along exactly the same lines as in the proof of Theorem 4 in [F5]. The second assertion follows from the observation that a $(\vec{\alpha},\vec{\beta},\gamma)$ shuffle followed by reversing the order of the cards is conjugate (by the longest length element in the symmetric group) to a $(\vec{\beta},\vec{\alpha},\gamma)$ shuffle. Alternatively, arguing as in the proof of Theorem 5 in [F5], one sees that the effect of reversing the cards on the cycle index of a $(\vec{\alpha},\vec{\beta},\gamma)$ shuffle is to get

$$\prod_{i,j} e^{\frac{((-u)^i x_i)^j}{ij} \sum_{d|i} \mu(d) (-\tilde{p}_{jd}(\vec{\alpha},\vec{\beta},\gamma))^{i/d}}.$$

□

Example 1 As a first application of Theorem 9, we derive an expression for the expected number of fixed points, generalizing the expressions in [DMP],[F5]. To get the generating function for fixed points, one sets $x_2 = x_3 = \dots = 1$ in the cycle index. Using the same trick as in [DMP] and [F5], the generating function simplifies to

$$\frac{1}{1-u} \frac{e^{ux\gamma}}{e^{ux}} \prod_{i \geq 1} \frac{1 - u\alpha_i}{1 - ux\alpha_i} \frac{1 + ux\beta_i}{1 + u\beta_i}.$$

Taking the derivative with respect to x and the coefficient of u^n , one sees that the expected number of fixed points is

$$\gamma + \sum_{j=1}^n \left[\sum_i (\alpha_i)^j + (-1)^{j+1} (\beta_i)^j \right].$$

This is exactly the sum of the first n extended power sum functions at the parameters $(\vec{\alpha},\vec{\beta},\gamma)$.

Example 2 We suppose that $\vec{\beta} = \vec{0}$ and that $\alpha_1 = \dots = \alpha_q = \frac{1-\gamma}{q}$. Then the cycle index simplifies to

$$\prod_{i \geq 1} \left(\frac{1}{1 - x_i \left(\frac{u(1-\gamma)}{q} \right)^i} \right)^{\frac{1}{i} \sum_{d|i} \mu(d) q^{i/d}} \prod_{i \geq 1} e^{\frac{u^i x_i (1-(1-\gamma)^i)}{i}}.$$

Of particular interest is the further specialization $q = 1$. Then the cycle index becomes

$$\frac{1}{1 - x_1 u(1 - \gamma)} \prod_{i \geq 1} e^{\frac{u^i x_i (1 - (1 - \gamma)^i)}{i}}.$$

Recall that a $(1/2, 0, 1/2)$ shuffle takes a binomial($n, 1/2$) number of cards, thoroughly mixes them, and then riffles them with the remaining cards. Example 3 on page 140 of [DFP] proves (in slightly different notation) that the convolution of k $(1/2, 0, 1/2)$ shuffles is the same as a $((1/2)^k, 0, 1 - (1/2)^k)$ shuffle. They conclude (in agreement with Theorem 8) that a $(1/2, 0, 1/2)$ shuffle takes $\log_2(n)$ steps to be mixed, as compared to $\frac{3}{2}\log_2(n)$ for ordinary riffle shuffles. They also establish a cut-off phenomenon. From the computation of Example 1 one sees that the expected number of fixed points also drops and that the mean mixes twice as fast.

As another example, consider a $(1 - 1/n, 0, 1/n)$ shuffle. Heuristically this is like top to random and [DFP] proves that the convergence rate is the same ($n \log(n)$ steps), which agrees with Theorem 8. From page 139 of [DFP], performing a $(1 - 1/n, 0, 1/n)$ shuffle k times is the same as performing a single $((1 - 1/n)^k, 0, 1 - (1 - 1/n)^k)$ shuffle. Example 1 gives a formula for the expected number of fixed points; it would be interesting to derive a cycle index for convolutions of top to random.

The approach of either [DMP] (method of moments) or [F5] (generating functions) can be used to prove the following limit result. For its statement, μ denotes the Moebius function of elementary number theory. Note that considerable simplifications take place when $q = 1$ (the interesting case) because $\sum_{d|i} \mu(d)$ is 1 if $i = 1$ and is 0 otherwise.

Corollary 3 *1. Fix u such that $0 < u < 1$. Choose a random deck size with probability of getting n equal to $(1 - u)u^n$. Let $N_i(\pi)$ be the number of i -cycles of π distributed as a $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffle where $\vec{\beta} = \vec{0}$ and $\alpha_1 = \dots = \alpha_q = \frac{1 - \gamma}{q}$. Then the random variables N_i are independent, where N_i is the convolution of a $\text{Poisson}((u^i(1 - (1 - \gamma)^i))/i)$ with $\frac{1}{i} \sum_{d|i} \mu(d)q^{i/d}$ many geometrics with parameter $(\frac{u(1 - \gamma)}{q})^i$.*

2. Let $N_i(\pi)$ be the number of i -cycles of π distributed as a $(\vec{\alpha}, \vec{\beta}, \gamma)$ shuffle where $\vec{\beta} = \vec{0}$ and $\alpha_1 = \dots = \alpha_q = \frac{1 - \gamma}{q}$. Then as $n \rightarrow \infty$ the random variables N_i are independent, where N_i is the convolution of a $\text{Poisson}((1 - (1 - \gamma)^i)/i)$ with $\frac{1}{i} \sum_{d|i} \mu(d)q^{i/d}$ many geometrics with parameter $(\frac{1 - \gamma}{q})^i$.

Example 3 As a final example, consider the case when $\alpha_1 = \dots = \alpha_q = \beta_1 = \dots = \beta_q = \frac{1}{2q}$ and all other parameters are 0. Theorems 3 and 9 imply that the distribution on RSK shape and

cycle index is the same as for the shuffles in Section 5 of [F5], though we do not see a simple reason why this should be so.

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References

- [AD] Aldous, D. and Diaconis, P., Shuffling cards and stopping times. *Amer. Math. Monthly* **93**, 333-348.
- [BaiRa] Baik, J. and Rains, E., Algebraic aspects of increasing subsequences. Preprint math.CO/9905083 at xxx.lanl.gov.
- [BayD] Bayer, D. and Diaconis, P., Trailing the dovetail shuffle to its lair. *Ann. Appl. Probab.* **2** (1992), 294-313.
- [BeRe] Berele, A. and Remmel, J., Hook flag characters and their combinatorics, *J. Pure Appl. Algebra*, **35** (1985), 225-245.
- [BorO] Borodin, A. and Olshanski, G., Z-measures on partitions, Robinson-Schensted-Knuth correspondence, and $\beta = 2$ random matrix ensembles. Preprint math.CO/9905189 at xxx.lanl.gov.
- [BotSi] Bottcher, A. and Silbermann, B., *Introduction to large truncated Toeplitz matrices*, Springer, 1999.
- [D] Diaconis, P., Group representations in probability and statistics. Institute of Mathematical Statistics Lecture Notes (1988) Volume 11.
- [DFP] Diaconis, P., Fill, J., and Pitman, J., Analysis of top to random shuffles, *Combin. Probab. Comput.*, **1** (1992), 135-155.
- [DMP] Diaconis, P., McGrath, M., and Pitman, J., Riffle shuffles, cycles, and descents. *Combinatorica* **15** (1995), 11-20.

- [F1] Fulman, J., The combinatorics of biased riffle shuffles. *Combinatorica* **18** (1998), 173-184.
- [F2] Fulman, J., Semisimple orbits of Lie algebras and card shuffling measures on Coxeter groups. *J. Algebra* **224** (2000), 151-165.
- [F3] Fulman, J., Affine shuffles, shuffles with cuts, the Whitehouse module, and patience sorting. *J. Algebra* **231** (2000), 614-639.
- [F4] Fulman, J., Applications of the Brauer complex: card shuffling, permutation statistics, and dynamical systems, to appear in *J. Algebra*.
- [F5] Fulman, J., Applications of symmetric functions to cycle and increasing subsequence structure after shuffles. Preprint math.CO/0102176 at xxx.lanl.gov.
- [G] Gessel, I., Symmetric functions and P -recursiveness, *J. Combin. Theory, Ser. A* **53** (1990), 257-285.
- [H] Hanlon, P., The action of S_n on the components of the Hodge decomposition of Hochschild homology. *Michigan Math. J.* **37** (1990), 105-124.
- [KV] Kerov, S. and Vershik, A., The characters of the infinite symmetric group and probability properties of the Robinson-Schensted-Knuth algorithm. *SIAM J. Algebraic Discrete Methods* **7** (1986), 116-124.
- [Ku] Kuperberg, G., Random words, quantum statistics, central limits, random matrices, Preprint math.PR/9909104 at xxx.lanl.gov.
- [J] Johansson, K., Discrete orthogonal polynomial ensembles and the Plancharel measure, Preprint math.CO/9906120 at xxx.lanl.gov.
- [La] Lalley, S., Cycle structure of riffle shuffles. *Ann. Probab.* **24** (1996), 49-73.
- [Mac] Macdonald, I., *Symmetric functions and Hall polynomials, 2nd edition*. Clarendon press, Oxford, 1995.
- [Ra] Rains, E., A mean identity for longest increasing subsequence problems. Preprint math.CO/0004082 at xxx.lanl.gov.
- [St1] Stanley, R., Generalized riffle shuffles and quasisymmetric functions. Preprint math.CO/9912025 at xxx.lanl.gov.

- [St2] Stanley. R., *Enumerative Combinatorics*. Vol. 2, Cambridge University Press, New York/Cambridge, 1999.
- [Th] Thoma, E., Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe, *Math. Z.* **85** (1964), 40-61.
- [TrWid] Tracy, C. and Widom, H., On the distribution of the lengths of the longest monotone subsequences in random words. Preprint math.CO/9904042 at xxx.lanl.gov.